

Zadatak 6.4 Nadite jednostavni verižni (neprekidni) razlomak od:

1. $\frac{2+\sqrt{5}}{3} = [1, 2, 2, 1, 12]$.

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$$\alpha = \frac{2 + \sqrt{5}}{3} \implies s_0 = 2, t_0 = 3, 3 \nmid (5 - 2^2) \implies$$

$$\alpha = \frac{6 + \sqrt{45}}{9} \implies s_0 = \mathbf{6}, t_0 = \mathbf{9}, 9 \mid (45 - 6^2)$$

$$a_0 = [\alpha_0] = \left[\frac{s_0 + \sqrt{45}}{t_0} \right] = \left[\frac{6 + \sqrt{45}}{9} \right] \stackrel{\alpha_0 \approx 1.4}{=} 1,$$

$$s_1 = a_0 t_0 - s_0 = 1 \cdot 9 - 6 = \mathbf{3}, t_1 = \frac{45 - s_1^2}{t_0} = \frac{45 - 3^2}{9} = \mathbf{4}$$

•

$$a_1 = [\alpha_1] = \left[\frac{s_1 + \sqrt{45}}{t_1} \right] = \left[\frac{3 + \sqrt{45}}{4} \right] \stackrel{\alpha_1 \approx 2.4}{=} 2,$$

$$s_2 = a_1 t_1 - s_1 = 2 \cdot 4 - 3 = \mathbf{5}, t_2 = \frac{45 - s_2^2}{t_1} = \frac{45 - 5^2}{5} = \mathbf{4}$$

•

$$a_2 = \left[\alpha_2 \right] = \left[\frac{s_2 + \sqrt{45}}{t_2} \right] = \left[\frac{5 + \sqrt{45}}{4} \right] \stackrel{\alpha_2 \simeq 2.9}{=} 2,$$

$$s_3 = a_2 t_2 - s_2 = 2 \cdot 4 - 5 = \mathbf{3}, \quad t_3 = \frac{45 - s_3^2}{t_2} = \frac{45 - 3^2}{4} = \mathbf{9}$$

•

$$a_3 = \left[\alpha_3 \right] = \left[\frac{s_3 + \sqrt{45}}{t_3} \right] = \left[\frac{3 + \sqrt{45}}{9} \right] \stackrel{\alpha_3 \simeq 1.1}{=} 1,$$

$$s_4 = a_3 t_3 - s_3 = 1 \cdot 9 - 3 = \mathbf{6}, \quad t_4 = \frac{45 - s_4^2}{t_3} = \frac{45 - 6^2}{9} = \mathbf{1}$$

•

$$a_4 = \left[\alpha_4 \right] = \left[\frac{s_4 + \sqrt{45}}{t_4} \right] = \left[\frac{6 + \sqrt{45}}{1} \right] \stackrel{\alpha_4 \simeq 12.7}{=} 12,$$

$$s_5 = a_4 t_4 - s_4 = 12 \cdot 1 - 6 = \mathbf{6}, \quad t_5 = \frac{45 - s_5^2}{t_4} = \frac{45 - 6^2}{1} = \mathbf{9}$$

•

$$(s_0, t_0) = (s_5, t_5) = (\mathbf{6}, \mathbf{9}) \implies j = 0, k = 5$$

$$\stackrel{r=5\text{-period}}{\implies} \alpha = [a_0, \dots, a_{j-1}, \overline{a_j, \dots, a_{k-1}}] =$$

$$= [\overline{a_0, \dots, a_{5-1}}] = [\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{12}].$$

2. $\sqrt{\frac{2c+1}{2c}} = [1, \overline{4c, 2}]$, $c \in \mathbb{N}$;

•

$$\alpha = \sqrt{\frac{2c+1}{2c}} = \frac{0 + \sqrt{2c(2c+1)}}{2c} \implies$$

$$s_0 = \mathbf{0}, \quad t_0 = \mathbf{2c}, \quad 2c \mid (2c(2c+1) - 0^2)$$

•

$$a_0 = \left\lfloor \alpha_0 \right\rfloor = \left\lfloor \frac{s_0 + \sqrt{2c(2c+1)}}{t_0} \right\rfloor = \left\lfloor \frac{0 + \sqrt{2c(2c+1)}}{2c} \right\rfloor$$

$$\alpha_0 = \frac{0 + \sqrt{2c(2c+1)}}{2c} = \sqrt{\frac{2c+1}{2c}} = \sqrt{1 + \frac{1}{2c}}$$

$$1 < \sqrt{1 + \frac{1}{2c}} < 2 \implies a_0 = \left\lfloor \alpha_0 \right\rfloor = 1$$

$$s_1 = a_0 t_0 - s_0 = 1 \cdot 2c - 0 = \mathbf{2c},$$

$$t_1 = \frac{2c(2c+1) - s_1^2}{t_0} = \frac{2c(2c+1) - (2c)^2}{2c} = \mathbf{1}$$

•

$$a_1 = \lfloor \alpha_1 \rfloor = \left\lfloor \frac{s_1 + \sqrt{2c(2c+1)}}{t_1} \right\rfloor = \left\lfloor \frac{2c + \sqrt{2c(2c+1)}}{1} \right\rfloor$$

$$\alpha_1 = 2c + \sqrt{2c(2c+1)}$$

$$2c = \sqrt{(2c)^2} < \sqrt{2c(2c+1)} < \sqrt{(2c+1)^2} = 2c+1 \implies$$

$$4c = 2c + 2c < 2c + \sqrt{2c(2c+1)} < 2c + 2c + 1 = 4c + 1 \implies$$

$$a_1 = \lfloor \alpha_1 \rfloor = 4c$$

$$s_2 = a_1 t_1 - s_1 = 4c \cdot 1 - 2c = \mathbf{2c},$$

$$t_2 = \frac{2c(2c+1) - s_2^2}{t_1} = \frac{2c(2c+1) - (2c)^2}{1} = \mathbf{2c}$$

•

$$a_2 = \lfloor \alpha_2 \rfloor = \left\lfloor \frac{s_2 + \sqrt{2c(2c+1)}}{t_2} \right\rfloor = \left\lfloor \frac{2c + \sqrt{2c(2c+1)}}{2c} \right\rfloor$$

$$\alpha_2 = \frac{2c + \sqrt{2c(2c+1)}}{2c} = 1 + \sqrt{\frac{2c+1}{2c}} = 1 + \sqrt{1 + \frac{1}{2c}}$$

$$2 < 1 + \sqrt{1 + \frac{1}{2c}} < 3 \implies$$

$$a_2 = \lfloor \alpha_2 \rfloor = \left\lfloor 1 + \sqrt{1 + \frac{1}{2c}} \right\rfloor = 2$$

$$s_3 = a_2 t_2 - s_2 = 2 \cdot 2c - 2c = \mathbf{2c},$$

$$t_2 = \frac{2c(2c+1) - s_3^2}{t_2} = \frac{2c(2c+1) - (2c)^2}{2c} = \mathbf{1}$$

•

$$(s_1, t_1) = (s_3, t_3) = (\mathbf{2c}, \mathbf{1}) \implies j = 1, k = 3$$

$$\begin{aligned} \xrightarrow{r=2\text{-period}} \alpha &= [a_0, \dots, a_{j-1}, \overline{a_j, \dots, a_{k-1}}] = \\ &= [a_0, \overline{a_1, a_2}] = [\mathbf{1}, \overline{\mathbf{4c}, \mathbf{2}}]. \end{aligned}$$

- $\sqrt{d^2 - d} = [d - 1, \overline{2, 2d - 2}]$, $d \in \mathbb{N}$, $d \geq 2$;

-

$$\alpha = \sqrt{d^2 - d} = \frac{0 + \sqrt{d^2 - d}}{1} \implies$$

$$s_0 = \mathbf{0}, \quad t_0 = \mathbf{1}, \quad 1 \mid ((d^2 - d) - 0^2)$$

-

$$a_0 = \lfloor \alpha_0 \rfloor = \left\lfloor \frac{s_0 + \sqrt{d^2 - d}}{t_0} \right\rfloor = \left\lfloor \frac{0 + \sqrt{d^2 - d}}{1} \right\rfloor$$

$$\alpha_0 = \sqrt{d^2 - d} = \sqrt{d(d - 1)}$$

$$d - 1 = \sqrt{(d - 1)^2} < \sqrt{d(d - 1)} < \sqrt{d^2} = d \implies$$

$$a_0 = \lfloor \alpha_0 \rfloor = d - 1$$

$$s_1 = a_0 t_0 - s_0 = (d - 1) \cdot 1 - 0 = \mathbf{d - 1},$$

$$t_1 = \frac{d(d - 1) - s_1^2}{t_0} = \frac{d(d - 1) - (d - 1)^2}{1} = \mathbf{d - 1}$$

•

$$a_1 = \lfloor \alpha_1 \rfloor = \left\lfloor \frac{s_1 + \sqrt{d^2 - d}}{t_1} \right\rfloor = \left\lfloor \frac{d - 1 + \sqrt{d^2 - d}}{d - 1} \right\rfloor$$

$$\alpha_1 = \frac{d - 1 + \sqrt{d^2 - d}}{d - 1} = 1 + \frac{\sqrt{d(d - 1)}}{d - 1} = 1 + \sqrt{\frac{d}{d - 1}} =$$

$$= 1 + \sqrt{1 + \frac{1}{d - 1}} \implies 2 < \alpha_1 < 3$$

$$a_1 = \lfloor \alpha_1 \rfloor = 2$$

$$s_2 = a_1 t_1 - s_1 = 2 \cdot (d - 1) - (d - 1) = \mathbf{d - 1},$$

$$t_2 = \frac{d(d - 1) - s_2^2}{t_1} = \frac{d(d - 1) - (d - 1)^2}{d - 1} = \mathbf{1}$$

•

$$a_2 = \left[\alpha_2 \right] = \left[\frac{s_2 + \sqrt{d^2 - d}}{t_2} \right] = \left[\frac{d - 1 + \sqrt{d^2 - d}}{1} \right]$$

$$\alpha_2 = d - 1 + \sqrt{d^2 - d} = d - 1 + \sqrt{d(d - 1)}$$

$$2d - 2 = d - 1 + \sqrt{(d - 1)^2} < d - 1 + \sqrt{d(d - 1)}$$

$$d - 1 + \sqrt{d(d - 1)} < d - 1 + \sqrt{d^2} = 2d - 1$$

$$\implies a_2 = \left[\alpha_2 \right] = 2d - 2$$

$$s_3 = a_2 t_2 - s_2 = (2d - 2) \cdot 1 - (d - 1) = \mathbf{d - 1},$$

$$t_3 = \frac{d(d - 1) - s_3^2}{t_2} = \frac{d(d - 1) - (d - 1)^2}{1} = \mathbf{d - 1}$$

•

$$(s_1, t_1) = (s_3, t_3) = (\mathbf{d - 1}, \mathbf{d - 1}) \implies j = 1, k = 3$$

$$\xrightarrow{r=2\text{-period}} \alpha = [a_0, \dots, a_{j-1}, \overline{a_j, \dots, a_{k-1}}] =$$

$$= [a_0, \overline{a_1, a_2}] = [\mathbf{d - 1}, \overline{\mathbf{2}, \mathbf{2d - 2}}].$$

Napomena:

- $\alpha = \frac{2+\sqrt{5}}{3} = [1, 2, 2, 1, 12]$ je reducirana kvadratna iracionalnost:

$$\alpha = \frac{2 + \sqrt{5}}{3} \simeq 1.412 > 1$$

$$\alpha' = \frac{2 - \sqrt{5}}{3} \simeq -0.0787 \implies -1 < \alpha' < 0$$

$\xRightarrow{\text{Tr. 6.11}}$ čisto periodski razvoj.

- $\alpha = \sqrt{6} = [2, \overline{2, 4}]$

Teorem 6.12:

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \dots, a_{r-1}, 2a_0}]$$

gdje je $a_0 = \left[\sqrt{d} \right]$, a niz a_1, a_2, \dots, a_{r-1} je centralno simetričan.

$$\sqrt{6} : a_0 = \left[\sqrt{6} \right] = 2, \quad r = 2 \implies [a_0; \overline{a_1, 2a_0}]$$

Zadatak 6.5 Nadite jednostavni verižni (neprekidni) razlomak od:

2. $\sqrt{29} = [5, \overline{2, 1, 1, 2, 10}]$.

•

$$\alpha = \sqrt{29} \implies s_0 = \mathbf{0}, t_0 = \mathbf{1}, a_0 = \left[\sqrt{29} \right] \stackrel{\alpha_0 \approx 5.4}{=} \mathbf{5}$$

•

$$s_1 = a_0 t_0 - s_0 = 5 \cdot 1 - 0 = \mathbf{5}, t_1 = \frac{29 - s_1^2}{t_0} = \frac{29 - 5^2}{1} = \mathbf{4}$$

$$a_1 = \left[\alpha_1 \right] = \left[\frac{s_1 + \sqrt{29}}{t_1} \right] = \left[\frac{5 + \sqrt{29}}{4} \right] \stackrel{\alpha_1 \approx 2.6}{=} \mathbf{2},$$

•

$$s_2 = a_1 t_1 - s_1 = 2 \cdot 4 - 5 = \mathbf{3}, t_2 = \frac{29 - s_2^2}{t_1} = \frac{29 - 3^2}{4} = \mathbf{5}$$

$$a_2 = \left[\alpha_2 \right] = \left[\frac{s_2 + \sqrt{29}}{t_2} \right] = \left[\frac{3 + \sqrt{29}}{5} \right] \stackrel{\alpha_2 \approx 1.7}{=} \mathbf{1}$$

•

$$s_3 = a_2 t_2 - s_2 = 1 \cdot 5 - 3 = \mathbf{2}, \quad t_3 = \frac{29 - s_3^2}{t_2} = \frac{29 - 3^2}{4} = \mathbf{5}$$

$$\implies t_2 = t_3 = 5 \xrightarrow{n=2} r = 2n + 1 = 2 \cdot 2 + 1 = 5$$

•

$$\sqrt{29} = [a_0; _, _, _, _, _] \implies [a_0; \overline{a_1, a_2, a_2, a_1, 2a_0}] \implies$$

$$\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$$

• **Napomena:** $\alpha = \sqrt{d^2 - d}$, $d \in \mathbb{N}$, $d \geq 2$;

$$\alpha = \sqrt{d^2 - d} \implies$$

$$s_0 = \mathbf{0}, \quad t_0 = \mathbf{1}, \quad a_0 = \lfloor \alpha_0 \rfloor = d - 1$$

$$s_1 = \mathbf{d} - \mathbf{1}, \quad t_1 = \mathbf{d} - \mathbf{1}, \quad a_1 = \lfloor \alpha_0 \rfloor = 2$$

$$s_2 = \mathbf{d} - \mathbf{1} \implies s_1 = s_2 = d - 1 \xrightarrow{n=1} r = 2n = 2 \cdot 1 = 2$$

$$\sqrt{d^2 - d} = [a_0; _, _, _] \implies [a_0; \overline{a_1, 2a_0}] \implies$$

$$\sqrt{d^2 - d} = [d - 1, \overline{2, 2d - 2}]$$

Primjer Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka 39.

Rješenje:

$(x, y, z) = (dx_1, dy_1, dz_1)$ i (x_1, y_1, z_1) primitivna

$$[d(m^2 - n^2)]^2 + (d \cdot 2mn)^2 = [d(m^2 + n^2)]^2.$$

$$(x_1, y_1, z_1) = (m^2 - n^2, 2mn, m^2 + n^2)$$

$$39 = 3 \cdot 13 \xrightarrow{\tau(39)=4} d = 1, 3, 13, 39$$

i) $d = 1$ (primitivne)

1) $m^2 + n^2 = 39 \implies$ (npr. m -paran) $n^2 = 3 \pmod{4} \implies$ nema;

2) $m^2 - n^2 = (m - n)(m + n) = 39$

$$39 = 39 \cdot 1 = 13 \cdot 3$$

2a) $m + n = 39, m - n = 1 \implies m = 20, n = 19 \implies$

$$z = 20^2 + 19^2 = 761$$

$$x = 20^2 - 19^2 = 39$$

$$y = 2 \cdot 20 \cdot 19 = 760$$

$$\implies (x, y, z) = (39, 760, 761)$$

Provjera: $39^2 + 760^2 = 579\,121 = 761^2$.

2b) $m + n = 13, m - n = 3 \implies m = 8, n = 5 \implies$

$$z = 8^2 + 5^2 = 89$$

$$x = 8^2 - 5^2 = 39$$

$$y = 2 \cdot 8 \cdot 5 = 80$$

$$\implies (x, y, z) = (39, 80, 89)$$

Provjera: $39^2 + 80^2 = 7921 = 89^2$.

i) d = 3

3) $m^2 + n^2 = 13 \implies (\text{npr. } m\text{-paran}) n^2 = 1 \pmod{4} \implies \text{dobro;}$

$$m^2 + n^2 = 13 \xrightarrow{m>n} 2n^2 \leq 13 \implies$$

$$n \leq \sqrt{\frac{13}{2}} = 2.5495 \implies n \leq 2 \implies$$

$$m^2 = 13 - n^2, n = 1, 2.$$

$$m^2 = 13 - 1^2 = 12 \implies \text{nema}$$

$$m^2 = 13 - 2^2 = 9 \implies m = 3, n = 2$$

$$z_1 = 3^2 + 2^2 = 13$$

$$x_1 = 3^2 - 2^2 = 5$$

$$y_1 = 2 \cdot 3 \cdot 2 = 12$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{15}, \mathbf{36}, \mathbf{39}).$$

Provjera: $15^2 + 36^2 = 1521 = 39^2$.

4) $m^2 - n^2 = (m - n)(m + n) = 13 \implies 13 = 13 \cdot 1$
 $m + n = 13, m - n = 1 \implies m = 7, n = 6 \implies$

$$z_1 = 7^2 + 6^2 = 85$$

$$x_1 = 7^2 - 6^2 = 13$$

$$y_1 = 2 \cdot 7 \cdot 6 = 84$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{39}, \mathbf{225}, \mathbf{252}).$$

Provjera: $39^2 + 225^2 = 52\,146 = 252^2$.

iii) d = 13

5) $m^2 + n^2 = 3 \implies (\text{npr. } m\text{-paran}) n^2 = 3 \pmod{4}$
 $\implies \text{nema;}$

$$\mathbf{6)} \quad m^2 - n^2 = (m - n)(m + n) = 3 \implies 3 = 3 \cdot 1$$
$$m + n = 3, m - n = 1 \implies m = 2, n = 1 \implies$$

$$z_1 = 2^2 + 2^2 = 8$$

$$x_1 = 2^2 - 1^2 = 3$$

$$y_1 = 2 \cdot 2 \cdot 1 = 4$$

$$\implies (x, y, z) = (13x_1, 13y_1, 13z_1) = (\mathbf{39}, \mathbf{52}, \mathbf{65}).$$

Provjera: $39^2 + 52^2 = 4225 = 65^2$.

d = 39 (nepotrebno)

$$\mathbf{7)} \quad m^2 + n^2 = 1 \implies m > n \implies m = 1, n = 0 \implies$$

nema;

$$\mathbf{8)} \quad m^2 - n^2 = (m - n)(m + n) = 1 \implies 1 = 1 \cdot 1$$
$$m + n = 1, m - n = 1 \implies m = 1, n = 0 \implies \text{nema};$$

Primjer Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka 12.

Rješenje:

$(x, y, z) = (dx_1, dy_1, dz_1)$ i (x_1, y_1, z_1) primitivna

$$[d(m^2 - n^2)]^2 + (d \cdot 2mn)^2 = [d(m^2 + n^2)]^2.$$

$$(x_1, y_1, z_1) = (m^2 - n^2, 2mn, m^2 + n^2)$$

$$12 = 3 \cdot 2^2 \xrightarrow{\tau(12)=6} d = 1, 2, 4, 3, 6, 12$$

i) d = 1 (primitivne)

$$\mathbf{1) } 2mn = 12 \implies mn = 6 \implies m = 6, n = 1 \text{ i } m = 3, n = 2 \implies$$

1a)

$$z = 6^2 + 1^2 = 37$$

$$x = 6^2 - 1^2 = 35$$

$$y = 2 \cdot 6 \cdot 1 = 12$$

$$\implies (x, y, z) = (\mathbf{35}, \mathbf{12}, \mathbf{37}).$$

$$\text{Provjera: } 35^2 + 12^2 = 1369 = 37^2.$$

1b)

$$z = 3^2 + 2^2 = 13$$

$$x = 3^2 - 2^2 = 5$$

$$y = 2 \cdot 3 \cdot 2 = 12$$

$$\implies (x, y, z) = (\mathbf{5}, \mathbf{12}, \mathbf{13}).$$

Provjera: $5^2 + 12^2 = 169 = 13^2$.

ii) d = 2

2) $2mn = 6 \implies mn = 3 \implies m = 3, n = 1$ ništa iste su parnosti;

iii) d = 4

3) $m^2 - n^2 = (m - n)(m + n) = 3 = 3 \cdot 1 \implies m + n = 3, m - n = 1 \implies m = 2, n = 1$

$$z = 2^2 + 1^2 = 5$$

$$x = 2^2 - 1^2 = 3$$

$$y = 2 \cdot 2 \cdot 1 = 4$$

$$\implies (x, y, z) = (4x_1, 4y_1, 4z_1) = (\mathbf{12}, \mathbf{16}, \mathbf{20}).$$

Provjera: $12^2 + 16^2 = 400 = 20^2$.

iv) $d = 3$

$$4) 2mn = 4 \implies mn = 2 \implies m = 2, n = 1 \implies$$

$$z = 2^2 + 1^2 = 5$$

$$x = 2^2 - 1^2 = 3$$

$$y = 2 \cdot 2 \cdot 1 = 4$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{9, 12, 15}).$$

Provjera: $9^2 + 12^2 = 225 = 15^2$.

v) $d = 6$

5) $2mn = 2 \implies mn = 1 \implies m = 1, n = 1 \implies$
ništa iste su parnosti;

Primjer Nađite sve Pitagorine trokute u kojima je jedna stranica jednaka 63.

Rješenje:

$(x, y, z) = (dx_1, dy_1, dz_1)$ i (x_1, y_1, z_1) primitivna

$$[d(m^2 - n^2)]^2 + (d \cdot 2mn)^2 = [d(m^2 + n^2)]^2.$$

$$(x_1, y_1, z_1) = (m^2 - n^2, 2mn, m^2 + n^2)$$

$$63 = 3^2 \cdot 7 \xrightarrow{\tau(63)=6} d = 1, 3, 9, 7, 21, 63$$

i) $d = 1$ (primitivne)

1) $m^2 + n^2 = 63 \implies$ (npr. m -paran) $n^2 = 3 \pmod{4} \implies$ nema;

2) $m^2 - n^2 = (m - n)(m + n) = 63$

$$63 = 63 \cdot 1 = 21 \cdot 3 = 9 \cdot 7$$

2a) $m + n = 63, m - n = 1 \implies m = 32, n = 31 \implies$

$$z = 32^2 + 31^2 = 1985$$

$$x = 32^2 - 31^2 = 63$$

$$y = 2 \cdot 32 \cdot 31 = 1984$$

$$\implies (x, y, z) = (\mathbf{63}, \mathbf{1984}, \mathbf{1985})$$

Provjera: $63^2 + 1984^2 = 3940\,225 = 1985^2$

2b) $m + n = 21, m - n = 3 \implies m = 12, n = 9 \implies$

$$z = 12^2 + 9^2 = 225$$

$$x = 12^2 - 9^2 = 63$$

$$y = 2 \cdot 12 \cdot 9 = 216$$

$$\implies (x, y, z) = (\mathbf{63}, \mathbf{216}, \mathbf{225})$$

Provjera: $63^2 + 216^2 = 50\,625 = 225^2$

2c) $m + n = 9, m - n = 7 \implies m = 8, n = 1 \implies$

$$z = 8^2 + 1^2 = 65$$

$$x = 8^2 - 1^2 = 63$$

$$y = 2 \cdot 8 \cdot 1 = 16$$

$$\implies (x, y, z) = (\mathbf{63}, \mathbf{16}, \mathbf{65}).$$

Provjera: $63^2 + 16^2 = 4225 = 65^2.$

ii) d = 3

1a) $m^2 + n^2 = 21 \implies$ (npr. m -paran) $n^2 = 1 \pmod{4}$
 \implies dobro;

$$2n^2 \leq m^2 + n^2 = 21 \implies n^2 \leq \frac{21}{2} = 10.5$$
$$\implies n \leq 3 \implies n = 1, 2, 3.$$

$$m^2 = 21 - 1 = 20 \implies \text{ništa};$$

$$m^2 = 21 - 4 = 17, \implies \text{ništa};$$

$$m^2 = 21 - 9 = 12 \implies \text{ništa}.$$

2) $m^2 - n^2 = (m - n)(m + n) = 21$

$$21 = 21 \cdot 1 = 7 \cdot 3$$

2a) $m + n = 21, m - n = 1 \implies m = 11, n = 10 \implies$

$$z_1 = 11^2 + 10^2 = 221$$

$$x_1 = 11^2 - 10^2 = 21$$

$$y_1 = 2 \cdot 11 \cdot 10 = 220$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{63}, \mathbf{660}, \mathbf{663})$$

Provjera: $63^2 + 660^2 = 439\,569 = 663^2$

$$\mathbf{2b)} \quad m + n = 7, m - n = 3 \implies m = 5, n = 2 \implies$$

$$z_1 = 5^2 + 2^2 = 29$$

$$x_1 = 5^2 - 2^2 = 21$$

$$y_1 = 2 \cdot 5 \cdot 2 = 20$$

$$\implies (x, y, z) = (3x_1, 3y_1, 3z_1) = (\mathbf{63}, \mathbf{60}, \mathbf{87})$$

$$\text{Provjera: } 63^2 + 60^2 = 7569 = 87^2$$

iii) d = 9

$$\mathbf{1a)} \quad m^2 + n^2 = 7 \implies (\text{npr. } m\text{-paran}) \quad n^2 = 3 \pmod{4} \\ \implies \text{ništa;}$$

$$\mathbf{2)} \quad m^2 - n^2 = (m - n)(m + n) = 7$$

$$7 = 21 \cdot 1$$

$$\mathbf{2a)} \quad m + n = 7, m - n = 1 \implies m = 4, n = 3 \implies$$

$$z_1 = 4^2 + 3^2 = 25$$

$$x_1 = 4^2 - 3^2 = 7$$

$$y_1 = 2 \cdot 4 \cdot 3 = 24$$

$$\implies (x, y, z) = (9x_1, 9y_1, 9z_1) = (\mathbf{63, 216, 225})$$

$$\text{Provjera: } 63^2 + 216^2 = 50\,625 = 225^2$$

iv) d = 21

1a) $m^2 + n^2 = 3 \implies$ (npr. m -paran) $n^2 = 3 \pmod{4}$
 \implies ništa;

2) $m^2 - n^2 = (m - n)(m + n) = 3$

$$3 = 3 \cdot 1$$

2a) $m + n = 3, m - n = 1 \implies m = 2, n = 1 \implies$

$$z_1 = 2^2 + 1^2 = 5$$

$$x_1 = 2^2 - 1^2 = 3$$

$$y_1 = 2 \cdot 2 \cdot 1 = 4$$

$$\implies (x, y, z) = (21x_1, 21y_1, 21z_1) = (\mathbf{63, 84, 105})$$

$$\text{Provjera: } 63^2 + 84^2 = 11\,025 = 105^2$$

Primjer Nađite najmanje rješenje u prirodnim brojevima Pellovih jednažbi $x^2 - 47y^2 = 1$ i $x^2 - 47y^2 = -1$ (ako postoje).

Rješenje:

- $$\alpha = \sqrt{47} = \frac{0 + \sqrt{47}}{1} \implies s_0 = \mathbf{0}, t_0 = \mathbf{1}, 1 \mid (47 - 0^2) \implies$$

$$a_0 = \left[\alpha_0 \right] = \left[\sqrt{47} \right] \stackrel{\alpha_0 \simeq 6.85}{=} \mathbf{6},$$

$$s_1 = a_0 t_0 - s_0 = 6 \cdot 1 - 0 = \mathbf{6}, t_1 = \frac{47 - s_1^2}{t_0} = \frac{47 - 6^2}{1} = \mathbf{11}$$

- $$a_1 = \left[\alpha_1 \right] = \left[\frac{s_1 + \sqrt{47}}{t_1} \right] = \left[\frac{6 + \sqrt{47}}{11} \right] \stackrel{\alpha_1 \simeq 1.2}{=} \mathbf{1},$$

$$s_2 = a_1 t_1 - s_1 = 1 \cdot 11 - 6 = \mathbf{5}, t_2 = \frac{47 - s_2^2}{t_1} = \frac{47 - 5^2}{11} = \mathbf{2}$$

-

$$a_2 = \left[\alpha_2 \right] = \left[\frac{s_2 + \sqrt{47}}{t_2} \right] = \left[\frac{5 + \sqrt{47}}{2} \right] \stackrel{\alpha_2 \simeq 5.9}{\equiv} \mathbf{5},$$

$$s_3 = a_2 t_2 - s_2 = 5 \cdot 2 - 5 = \mathbf{5}, \quad t_3 = \frac{47 - s_3^2}{t_2} = \frac{47 - 5^2}{2} = \mathbf{11}$$

-

$$\implies s_2 = s_3 = 5 \xrightarrow{n=2} \mathbf{r} = 2n = 2 \cdot 2 = \mathbf{4}$$

$$\sqrt{47} = [a_0; _, _, _, _] \implies [a_0; \overline{a_1, a_2, a_1, 2a_0}] \implies$$

$$\sqrt{47} = [6, \overline{1, 5, 1, 12}]$$

- Kako je $r = 4$ paran, onda jednačba $x^2 - 47y^2 = -1$ nema rješenja, a sva rješenja u prirodnim brojevima od $x^2 - 47y^2 = 1$ su dana sa $x = p_{4k-1}$, $y = q_{4k-1}$, $k \in \mathbb{N}$.

- Najmanje $(x, y) = (p_3, q_3)$;

- Traženje konvergenti, tj. rješenja u prirodnim brojevima:

$$p_n = a_n p_{n-1} + p_{n-2}, \quad p_0 = a_0, \quad p_1 = a_0 a_1 + 1;$$

$$q_n = a_n q_{n-1} + q_{n-2}, \quad q_0 = 1, \quad q_1 = a_1.$$

n	0	1	2	3
a_n	6	1	5	1
p_n	6	7	41	48
q_n	1	1	6	7

- Najmanje $(x, y) = (48, 7)$.