4.3 Teorija o poljima

- U ovomu ćemo odjeljku ćemo dati nekoliko primjena skalarne i vektorske analize u prostoru $\mathbb{R}^3$;

- Slučaj je naročito važan jer se u njemu opisuje naš fizički (tvarni) svijet.

- Oznake ćemo prilagoditi onima tradicionalnim što dolaze iz fizike.

Skalarno i vektorsko polje

Oznake:

- Za svaku točku T u prostoru $\mathbb{E}$ neka $\overrightarrow{\mathbb{V}}(T)$ označuje skup svih radijus-vektora $\overrightarrow{\boldsymbol{r}}_P$ (usmjerenih dužina $\overrightarrow{TP}$) svih točaka P u prostoru $\mathbb{E}$ s obzirom na točku T;

- $\overrightarrow{\mathbb{V}} \equiv \cup\{\overrightarrow{\mathbb{V}}(T) \mid T$ točka u $\mathbb{E}\}$.

Definicija 4.10 Neka je Ω skup točaka u $\mathbb{E}$. Svaku funkciju $U : \Omega \to \mathbb{R}$ nazivamo **skalarnim poljem**, a svaku funkciju $\overrightarrow{\boldsymbol{V}} : \Omega \to \overrightarrow{\mathbb{V}}$ - **vektorskim poljem** na danom skupu Ω.

- Drugim riječima, na točkovnom skupu Ω je zadano skalarno (vektorsko) polje čim je svakoj točki $T \in \Omega$ pridijeljen točno jedan broj $U(T) \in \mathbb{R}$ (radijus-vektor $\overrightarrow{\boldsymbol{r}}_P \in \overrightarrow{\mathbb{V}}(T)$).

- Primijetimo da definicija skalarnog i vektorskog polja ne ovisi o koordinatizaciji prostora $\mathbb{E}$. Međutim, u svakom koordinatnom sustavu $\mathcal{S}$ u $\mathbb{E}$, svako skalarno polje $U : \Omega \to \mathbb{R}$ posve određeno nekom funkcijom

$$f : D \equiv \Omega_{\mathcal{S}} \to \mathbb{R},\ \Omega_{\mathcal{S}} \subseteq \mathbb{E}_{\mathcal{S}},\ f(T_{\mathcal{S}}) = U(T)$$

  (indeks podsjeća na uvedeni koordinatni sustav).

- Slično vrijedi za svako vektorsko polje.

Ako u $\mathbb{E}$ uvedemo Kartezijev desni pravokutni koordinatni sustav

$$\mathcal{S} = \left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right),$$

prostor $\mathbb{E}$ se identificira s $\mathbb{R}^3$ pa se svako skalarno polje $U : \Omega \to \mathbb{R}$ opisuje nekom (skalarnom) funkcijom

$$f : D \to \mathbb{R},\ f(x, y, z) = U(T),$$

a svako vektorsko polje $\overrightarrow{\boldsymbol{V}} : \Omega \to \overrightarrow{\mathbb{V}}$ nekom (vektorskom) funkcijom

$$\overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}^3,\ \overrightarrow{\boldsymbol{w}}(x, y, z) = \overrightarrow{\boldsymbol{V}}(T),$$

pri čemu su x, y i z koordinate točke $T \in \Omega$ u sustavu $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$, tj. $T \equiv (x, y, z) \in D \subseteq \mathbb{R}^3$. Pritom je, dakle,

$$\overrightarrow{\boldsymbol{V}}(T) = w_x(x, y, z)\overrightarrow{\boldsymbol{i}} + w_y(x, y, z)\overrightarrow{\boldsymbol{j}} + w_z(x, y, z)\overrightarrow{\boldsymbol{k}},$$

gdje su $w_x, w_y, w_z : D \to \mathbb{R}$ koordinatne funkcije od $\overrightarrow{\boldsymbol{w}}$, tj. $\overrightarrow{\boldsymbol{w}} = (w_x, w_y, w_z)$.

- Ponekad nećemo, jednostavnosti radi, praviti strogu jezičnu razliku između ”polja” i ”funkcije”.

- Osim toga, najčešće nećemo međusobno formalno razlikovati ni skupove $\overrightarrow{\mathbb{V}}(T)$, $T \in \Omega$, nego ćemo svakoga od njih poistovjetiti s $\overrightarrow{\mathbb{V}}(O)$, gdje će O biti ishodište Kartezijeva desnog pravokutnog sustava $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$.

Primjer 10

(1) Neka je Ω Zemljina kruta površina zamišljena kao točkovni skup, te neka je, za svaki $T \in \Omega$, $U(T)$ nadmorska visina, odnosno, podmorska dubina te točke (s obzirom na dogovorenu nultu (nadmorsku) razinu). Tada je $U : \Omega \to \mathbb{R}$ skalarno polje - ”reljefni globus”;

(2) Neka je $f(x, y, z) = \frac{z}{x^2+y^2}$. Tada je funkcijom

$$f : \mathbb{R}^2 \backslash \{(0, 0, z) \mid z \in \mathbb{R}\} \to \mathbb{R}, \quad (x, y, z) \mapsto f(x, y, z),$$

na pripadnom području posve zadano skalarno polje

$$T \mapsto U(T) = \frac{z}{x^2 + y^2},$$

pri čemu su $x, y,$ i z Kartezijeve koordinate točke T u odabranom sustavu $\left(O; \vec{\boldsymbol{i}}, \vec{\boldsymbol{j}}, \vec{\boldsymbol{k}}\right)$.

Primjer 11

(1) Neka Ω predstavlja Zemljin zračni omotač kao točkovni skup, te neka je, za svaki $T \in \Omega$, $\vec{\boldsymbol{V}}(T)$ brzina zračnoga strujanja u toj točki (vjetrovni vektor) u odabranom trenutku. Tada je $\vec{\boldsymbol{V}} : \Omega \to \vec{\mathbb{V}}$ vektorsko polje brzine vjetrova u atmosferi u odabranom trenutku;

(2) Neka je vektorska funkcija $\vec{\boldsymbol{w}} : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}^3$ zadana koordinatnim funkcijama

$$w_x(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}, \ w_y(x,y,z) = \frac{y}{\sqrt{x^2+y^2+z^2}},$$

$$w_z(x,y,z) = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

Tada je na $\Omega \equiv \mathbb{R}^3 \setminus \{(0,0,0)\}$ zadano vektorsko polje

$$T \mapsto \overrightarrow{\boldsymbol{V}}(T) = w(x,y,z) =$$

$$w_x(x,y,z)\overrightarrow{\boldsymbol{i}} + w_y(x,y,z)\overrightarrow{\boldsymbol{j}} + w_z(x,y,z)\overrightarrow{\boldsymbol{k}} =$$

$$= \frac{x\overrightarrow{\boldsymbol{i}} + y\overrightarrow{\boldsymbol{j}} + z\overrightarrow{\boldsymbol{k}}}{\sqrt{x^2+y^2+z^2}},$$

pri čemu su x, y i z Kartezijeve koordinate točke T u odabranomu sustavu $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$.

Primijetimo da se radi o polju svih jediničnih radijus-vektora,

$$T \mapsto \overrightarrow{\boldsymbol{r}}_0(T) = \frac{\overrightarrow{OT}}{\left\|\overrightarrow{OT}\right\|}, \quad T \neq O = (0,0,0),$$

s obzirom na dano ishodište O.)

Promatrajmo skalarno polje $U : \Omega \to \mathbb{R}$. Uvjet $U(T) = c$ (konstanta) određuje tzv. razinsku ili **ekvipotencijalnu plohu** skalarnoga polja U.

Ako je Ω točkovni (pod)skup neke plohe, uvjet $U(T) = c$ određuje tzv. **ekvipotencijalnu krivulju** skalarnoga polja U.

Jednako govorimo u slučaju skalarnog polja $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$ ($D \subseteq \mathbb{R}^2$).

Tako u Primjeru 10 (1) krivulje *izobare* i *izohipse*, a u Primjeru 10 (2) - rotacijske paraboloide $z = c(x^2 + y^2)$, $c \in \mathbb{R}$.

U vektorskom polju $\overrightarrow{\boldsymbol{V}} : \Omega \to \overrightarrow{\mathbb{V}}$ je zanimljivo promatrati tzv. **strujnice** (**silnice** ili **vektorske linije**), što se definiraju kao krivulje kojima se tangente podudaraju s pravcima vektorskoga polja $\overrightarrow{\boldsymbol{V}}$ u svakoj točki $T \in \Omega$. Prema tomu, ako je vektorsko polje $\overrightarrow{\boldsymbol{V}}$ zadano funkcijom $\overrightarrow{\boldsymbol{w}} = (w_x, w_y, w_z)$, onda mu se strujnice određuju iz jednadžbe

$$\frac{d\overrightarrow{\boldsymbol{r}}}{dt} = c\overrightarrow{\boldsymbol{w}}, \quad c \in \mathbb{R},$$

pri čemu se traži $\overrightarrow{\boldsymbol{r}} = (x, y, z)$, tj.

$$\overrightarrow{\boldsymbol{r}}(t) = \phi(t)\overrightarrow{\boldsymbol{i}} + \psi(t)\overrightarrow{\boldsymbol{j}} + \chi(t)\overrightarrow{\boldsymbol{k}}.$$

Obično se eliminacijom parametra t dobva sustav diferencijalnih jednadžaba

$$\frac{dx}{w_x} = \frac{dy}{w_y} = \frac{dz}{w_z}.$$

Tako se u Primjeru 11 (2) dobivamo sustav

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z},$$

što daje $y = c_1 x$ i $z = c_2 x$, odnosno,

$$\frac{x}{1} = \frac{y}{c_1} = \frac{z}{c_2}.$$

Radi se, dakle o skupu svih pravaca u prostoru koji prolaze odabranim ishodištem $O = (0, 0, 0)$.

Napomena Skalarno i vektorsko polje se mogu definirati općenitije u smislu da se uvede i ovisnost o vremenu. Takva polja nazivamo **nestacionarnima**, za razliku od prije definiranih koja onda nazivamo **stacionarnima**.

Primjer 12

- Mjerimo li tako u temperaturu tijekom nekog vremenskog intervala, dobivamo primjer nestacionarnoga skalarnog polja, dok su u Primjeru 10 primjeri stacionarnog skalarnog polja.

- Nadalje, mjerimo li u Primjeru 11 (a) brzinu zračnoga strujanja tijekom nekog vremenskog intervala, dobivamo primjer nestacionarnoga vektorskog polja, dok je Primjer 11 (2) primjer stacionarnog vektorskog polja.

Definicija 4.11 Reći ćemo da je skalarno polje $U : \Omega \to \mathbb{R}$ **neprekidno** *(***diferencijabilno***)* ako je njegov predstavnik $f : D \to \mathbb{R}$, u provokutnom koordinatnom sustavu $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$, neprekidna *(*diferencijabilna*)* funkcija.

Slično, reći ćemo da je vektorsko polje $\overrightarrow{\boldsymbol{V}} : \Omega \to \overrightarrow{\mathbb{V}}$ **neprekidno** *(***diferencijabilno***)* ako je pripadna vektorska funkcija $\overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}^3$ neprekidna *(*diferencijabilna*).*

Napomena Primijetimo da prethodna definicija nije apriori korektna. Trebalo bi, naime, uz nju dokazati da su neprekidnost i derivabilnost invarijante s obzirom na izbor pravokutnog koordinatnog sustava u prostoru $\mathbb{E}$, odnosno, da ne ovise o izboru predstavljajuće funkcije.

Gradijent, divergencija i rotacija

- Ovdje ćemo definirati tri linearna operatora, neophodna za matematičko opisivanje temeljnih fizičkih zakona tvarnoga svijeta u kojemu živimo. Radi se u stvari o jednom operatoru djelovanje kojega se očituje na tri načina - ovisno o objektima na koje djeluje.

- Jednostavnosti radi, skalarna i vektorska polja ćemo odmah zadavati njihovim predstavnicima, tj. skalarnim i vektorskim funkcijama u Kartezijevu desnom pravokutnom koordinatnom sustavu $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$ u $\mathbb{E} \equiv \mathbb{R}^3$.

Definicija 4.12 Neka su $f : D \to \mathbb{R}$ i $\overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, redom skalarno i vektorsko polje, te neka su oba diferencijabilna.

- **Gradijentom** skalarnoga polja f nazivamo vektorsko polje $\operatorname{grad} f : D \to \mathbb{R}^3$,

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

tj.

$$\operatorname{grad} f(x, y, z) =$$

$$\frac{\partial f(x,y,z)}{\partial x} \overrightarrow{\boldsymbol{i}} + \frac{\partial f(x,y,z)}{\partial y} \overrightarrow{\boldsymbol{j}} + \frac{\partial f(x,y,z)}{\partial z} \overrightarrow{\boldsymbol{k}}, \quad (x, y, z) \in D.$$

- **Divergencijom** vektorskoga polja $\overrightarrow{\boldsymbol{w}} = (w_x, w_y, w_z)$ nazivamo skalarno polje $\operatorname{div} \overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}$,

$$\operatorname{div} \overrightarrow{\boldsymbol{w}} = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z},$$

tj.

$$\operatorname{div} \overrightarrow{\boldsymbol{w}}(x, y, z) =$$

$$\frac{\partial w_x(x,y,z)}{\partial x} + \frac{\partial w_y(x,y,z)}{\partial y} + \frac{\partial w_z(x,y,z)}{\partial z}, \quad (x, y, z) \in D.$$

- **Rotacijom** vektorskoga polja $\overrightarrow{\boldsymbol{w}}$ nazivamo vektorsko polje $\operatorname{rot}\overrightarrow{\boldsymbol{w}}: D \to \mathbb{R}^3$,

$$\operatorname{rot}\overrightarrow{\boldsymbol{w}} = \left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z},\frac{\partial w_x}{\partial z}-\frac{\partial w_z}{\partial x},\frac{\partial w_y}{\partial x}-\frac{\partial w_x}{\partial y}\right),$$

tj.

$$\operatorname{rot}\overrightarrow{\boldsymbol{w}}(x,y,z) = \left(\frac{\partial w_z(x,y.z)}{\partial y} - \frac{\partial w_y(x,y,z)}{\partial z}\right)\overrightarrow{\boldsymbol{i}} +$$

$$+\left(\frac{\partial w_x(x,y,z)}{\partial z} - \frac{\partial w_z(x,y,z)}{\partial x}\right)\overrightarrow{\boldsymbol{j}} +$$

$$+\left(\frac{\partial w_y(x,y,z)}{\partial x} - \frac{\partial w_x(x,y,z)}{\partial y}\right)\overrightarrow{\boldsymbol{k}}, \quad (x,y,z) \in D.$$

Primijetimo da $\operatorname{rot}\overrightarrow{\boldsymbol{w}}$ dopušta formalni zapis u obliku determinante:

$$\operatorname{rot}\overrightarrow{\boldsymbol{w}} = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_x & w_y & w_z \end{vmatrix}.$$

Uočimo:

- gradijent je funkcija definirana na skupu svih diferencijabilnih skalarnih polja s vrijednostima u skupu diferencijabilnih vektorskih polja;

- divergencija je funkcija definirana na skupu svih diferencijabilnih vektorskih polja s vrijednostima u skupu skalarnih polja;

- rotacija je funkcija definirana na skupu svih diferencijabilnih vektorskih polja s vrijednostima u istomu skupu.

Primjer 13

(1) Odredimo gradijent skalarnoga polja $(x, y, z) \mapsto f(x, y, z) = xy^2z^3$.

$$\operatorname{grad} f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)(x, y, z) =$$

$$\left(\frac{\partial f\left(xy^2z^3\right)}{\partial x}, \frac{\partial f\left(xy^2z^3\right)}{\partial y}, \frac{\partial f\left(xy^2z^3\right)}{\partial z}\right) =$$

$$\left(y^2z^3, 2xyz^3, 3xy^2z^2\right)$$

(2) Odredimo divergenciju vektorskoga polja $(x, y, z) \mapsto \overrightarrow{\boldsymbol{w}}(x, y, z) = \big(2x, xy^2, xz^2\big)$.

$$\operatorname{div} \overrightarrow{\boldsymbol{w}}(x, y, z) = \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right)(x, y, z) =$$

$$\left(\frac{\partial(2x)}{\partial x} + \frac{\partial(xy^2)}{\partial y} + \frac{\partial(xz^2)}{\partial z}\right) = 2 + 2xy - 2xz$$

(3) Odredimo rotaciju vektroskoga polja $\overrightarrow{\boldsymbol{w}}$ iz (2).

$$\operatorname{rot} \overrightarrow{\boldsymbol{w}}(x, y, z) = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_x & w_y & w_z \end{vmatrix} (x, y, z) = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & xy^2 & xz^2 \end{vmatrix} =$$

$$\left(\frac{\partial\left(xz^2\right)}{\partial y} - \frac{\partial\left(xy^2\right)}{\partial z}\right)\overrightarrow{\boldsymbol{i}} + \left(\frac{\partial\left(2x\right)}{\partial z} - \frac{\partial\left(xz^2\right)}{\partial x}\right)\overrightarrow{\boldsymbol{j}} +$$

$$+\left(\frac{\partial\left(xy^2\right)}{\partial x} - \frac{\partial\left(2x\right)}{\partial y}\right)\overrightarrow{\boldsymbol{k}} =$$

$$0\,\overrightarrow{\boldsymbol{i}} + \left(-z^2\right)\overrightarrow{\boldsymbol{j}} + \left(y^2\right)\overrightarrow{\boldsymbol{k}} = \left(0, -z^2, y^2\right)$$

Gradijent, divergencija i rotacija mogu opisati samo jednim operatorom.

Znakom ∇ (čitamo: nabla) označimo tzv. **Hamiltonov diferencijalni operator** - formalni vektor

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \overrightarrow{\boldsymbol{i}}\frac{\partial}{\partial x} + \overrightarrow{\boldsymbol{j}}\frac{\partial}{\partial y} + \overrightarrow{\boldsymbol{k}}\frac{\partial}{\partial z}.$$

Gledamo li na ∇ kao na operator definiran na vektorskom prostoru svih diferencijabilnih skalarnih (vektorskih) polja, pojavljuju se, sljedeće mogućnosti:

$$\begin{aligned}
\nabla f &= \operatorname{grad} f, && \big(\text{djelovanje operatora } \nabla \text{ na skalarno polje } f\big) \\
\nabla \circ \overrightarrow{\boldsymbol{w}} &= \operatorname{div} \overrightarrow{\boldsymbol{w}}, && (\text{formalni skalarni umnožak } \nabla \text{ i vektorskog polja } \overrightarrow{\boldsymbol{w}}) \\
\nabla \times \overrightarrow{\boldsymbol{w}} &= \operatorname{rot} \overrightarrow{\boldsymbol{w}}, && \big(\text{formalni vektorski umnožak } \nabla \text{ i vektorskog polja } \overrightarrow{\boldsymbol{w}}\big);
\end{aligned}$$

Važna je činjenica da je operator ∇ linearan u svakoj od navedenih interpretacija, tj. da vrijedi ovaj teorem:

Teorem 4.13 Nabla je linearni operator, tj. za bilo koja dva diferencijabilna skalarna polja $f, g : D \to \mathbb{R}$, bilo koja dva diferencijabilna vektorska polja $\overrightarrow{\boldsymbol{w}}, \overrightarrow{\boldsymbol{u}} : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, i bilo koja dva broja $\lambda, \mu \in \mathbb{R}$ vrijedi:

1. $\nabla(\lambda f + \mu g) = \lambda \nabla f + \mu \nabla g$;

2. $\nabla \circ \left(\lambda \overrightarrow{\boldsymbol{w}} + \mu \overrightarrow{\boldsymbol{u}}\right) = \lambda \left(\nabla \circ \overrightarrow{\boldsymbol{w}}\right) + \mu \left(\nabla \circ \overrightarrow{\boldsymbol{u}}\right)$;

3. $\nabla \times \left(\lambda \overrightarrow{\boldsymbol{w}} + \mu \overrightarrow{\boldsymbol{u}}\right) = \lambda \left(\nabla \times \overrightarrow{\boldsymbol{w}}\right) + \mu \left(\nabla \times \overrightarrow{\boldsymbol{u}}\right)$.

Dokaz: Dokažimo npr. tvrdnju (3):

$$\nabla \times \left(\lambda \overrightarrow{\boldsymbol{w}} + \mu \overrightarrow{\boldsymbol{u}}\right) = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \lambda w_x + \mu u_x & \lambda w_y + \mu u_y & \lambda w_z + \mu u_z \end{vmatrix} =$$

$$= \lambda \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_x & w_y & w_z \end{vmatrix} + \mu \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} =$$

$$= \lambda \left(\nabla \times \overrightarrow{\boldsymbol{w}}\right) + \mu \left(\nabla \times \overrightarrow{\boldsymbol{u}}\right).$$

Teorem 4.14 Neka su $f, g : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$, diferencijabilna skalarna polja, c_r konstantno skalarno polje, $\phi : Y \to \mathbb{R}$, $f[D] \subseteq Y \subseteq \mathbb{R}$, diferencijabilna funkcija i $\lambda, \mu \in \mathbb{R}$. Tada je

1. $\operatorname{grad} c_r = \overrightarrow{\mathbf{0}}$;
2. $\operatorname{grad}(\lambda f + \mu g) = \lambda \operatorname{grad} f + \mu \operatorname{grad} g$;
3. $\operatorname{grad}(f \cdot g) = g \operatorname{grad} f + f \operatorname{grad} g$;
4. $\operatorname{grad}\left(\dfrac{f}{g}\right) = \dfrac{g \operatorname{grad} f - f \operatorname{grad} g}{g^2}$ (čim je $g(x) \neq 0$);
5. $\operatorname{grad}(\phi \circ f) = \phi' \operatorname{grad} f$.

Dokaz: Tvrdnje su izravna posljedica definicije gradijenta. Primijetimo da je tvrdnja (2) isto što i Teorem 4.13 (1).

Teorem 4.15 Neka su $\overrightarrow{\boldsymbol{w}}, \overrightarrow{\boldsymbol{u}} : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, diferencijabilna vektorska polja, $f, g : D \to \mathbb{R}$ diferencijabilna skalarna polja, $\overrightarrow{\boldsymbol{c}}$ konstantno vektorsko polje i $\lambda, \mu \in \mathbb{R}$. Tada je

1. $\operatorname{div} \overrightarrow{\boldsymbol{c}} = \mathbf{0}$;

2. $\operatorname{div}\left(\lambda \overrightarrow{\boldsymbol{w}} + \mu \overrightarrow{\boldsymbol{u}}\right) = \lambda \operatorname{div} \overrightarrow{\boldsymbol{w}} + \mu \operatorname{div} \overrightarrow{\boldsymbol{u}}$;

3. $\operatorname{div}\left(\overrightarrow{\boldsymbol{w}} \times \overrightarrow{\boldsymbol{u}}\right) = \left(\operatorname{rot} \overrightarrow{\boldsymbol{w}}\right) \circ \overrightarrow{\boldsymbol{u}} - \overrightarrow{\boldsymbol{w}} \circ \left(\operatorname{rot} \overrightarrow{\boldsymbol{u}}\right)$;

4. $\operatorname{div}\left(f \cdot \overrightarrow{\boldsymbol{w}}\right) = (\operatorname{grad} f) \circ \overrightarrow{\boldsymbol{w}} + f \cdot \left(\operatorname{div} \overrightarrow{\boldsymbol{w}}\right)$;

5. $\operatorname{div}(f \cdot \operatorname{grad} g) = (\operatorname{grad} f) \circ (\operatorname{grad} g) + f \triangle g$
   (g dvaput diferencijabilno), pri čemu je

$$\Delta \equiv \operatorname{div} \operatorname{grad} = \nabla \circ \nabla$$

   tzv. **Laplaceov diferencijalni operator** (delta), tj.

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2};$$

6. $\operatorname{div}(\operatorname{rot} \overrightarrow{\boldsymbol{w}}) = \overrightarrow{\mathbf{0}}$ ($\overrightarrow{\boldsymbol{w}}$ dvaput diferencijabilno).

Dokaz: Tvrdnje slijede neposredno iz definicije divergencije. Dokažimo npr. tvrdnju (4)

$$\operatorname{div}(f \cdot \overrightarrow{\boldsymbol{w}}) = \frac{\partial(f w_x)}{\partial x} + \frac{\partial(f w_y)}{\partial y} + \frac{\partial(f w_z)}{\partial z} =$$

$$\frac{\partial f}{\partial x} w_x + f \frac{\partial w_x}{\partial x} + \frac{\partial f}{\partial y} w_y + f \frac{\partial w_y}{\partial y} + \frac{\partial f}{\partial z} w_z + f \frac{\partial w_z}{\partial z} =$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \circ (w_x, w_y, w_z) + f \cdot \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}\right) =$$

$$(\operatorname{grad} f) \circ \overrightarrow{\boldsymbol{w}} + f \cdot \operatorname{div} \overrightarrow{\boldsymbol{w}}.$$

Primijetimo da se tvrdnja 2. podudara s onom iz Teorema 1.13, 2. Tvrdnja 5. slijedi iz 4. čim je $\overrightarrow{\boldsymbol{w}} = \operatorname{grad} g$. Napokon, u dokazu tvrdnje 6. treba primijeniti Schwarzov teorem.

Teorem 4.16 Pod istim pretpostavkama kao u Teoremu 4.16 vrijedi:

1. $\operatorname{rot}\overrightarrow{\boldsymbol{c}} = \overrightarrow{\mathbf{0}}$;

2. $\operatorname{rot}\left(\lambda\overrightarrow{\boldsymbol{w}} + \mu\overrightarrow{\boldsymbol{u}}\right) = \lambda\operatorname{rot}\overrightarrow{\boldsymbol{w}} + \mu\operatorname{rot}\overrightarrow{\boldsymbol{u}}$;

3. $\operatorname{rot}\left(\overrightarrow{\boldsymbol{w}} \times \overrightarrow{\boldsymbol{u}}\right) = \left(\operatorname{div}\overrightarrow{\boldsymbol{u}}\right)\overrightarrow{\boldsymbol{w}} - \left(\operatorname{div}\overrightarrow{\boldsymbol{w}}\right)\overrightarrow{\boldsymbol{u}} + \left(\overrightarrow{\boldsymbol{u}} \circ \nabla\right)\overrightarrow{\boldsymbol{w}} - \left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right)\overrightarrow{\boldsymbol{u}}$, pri čemu su
$$\overrightarrow{\boldsymbol{w}} \circ \nabla \equiv w_x\frac{\partial}{\partial x} + w_y\frac{\partial}{\partial y} + w_z\frac{\partial}{\partial z}$$
i, slično, $\left(\overrightarrow{\boldsymbol{u}} \circ \nabla\right)$ novi diferencijalni operatori;

4. $\operatorname{rot}\left(f \cdot \overrightarrow{\boldsymbol{w}}\right) = (\operatorname{grad} f) \times \overrightarrow{\boldsymbol{w}} - f\operatorname{rot}\overrightarrow{\boldsymbol{w}}$;

5. $\operatorname{rot}(f \cdot \operatorname{grad} f) = (\operatorname{grad} f) \times (\operatorname{grad} g)$, i posebice, $\operatorname{rot}(\operatorname{grad} f) = \overrightarrow{\mathbf{0}}$;

6. $\operatorname{rot}\left(\operatorname{rot}\overrightarrow{\boldsymbol{w}}\right) = \operatorname{grad}\left(\operatorname{div}\overrightarrow{\boldsymbol{w}}\right) - \Delta\overrightarrow{\boldsymbol{w}}$
($\overrightarrow{\boldsymbol{w}}$ dvaput diferencijabilno*)*, pri čemu Laplaceov operator Δ treba primijeniti na svaku koordinatnu funkciju od $\overrightarrow{\boldsymbol{w}}$.

Napomena: Diferencijalni operator

$$\overrightarrow{\boldsymbol{w}} \circ \nabla \equiv w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + w_z \frac{\partial}{\partial z}$$

shvaćamo kao "skalar", pa on može djelovati i na vektorsko i skalarno polje. Imamo:

- ako je f diferencijabilno skalarno polje onda je

$$\left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right) f \equiv \left(w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + w_z \frac{\partial}{\partial z}\right) f =$$

$$= w_x \frac{\partial f}{\partial x} + w_y \frac{\partial f}{\partial y} + w_z \frac{\partial f}{\partial z} = \overrightarrow{\boldsymbol{w}} \circ \operatorname{grad} f$$

- ako je $\overrightarrow{\boldsymbol{u}} \equiv (u_x, u_y, u_z)$ diferencijabilno vektorsko polje onda je

$$\left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right) \overrightarrow{\boldsymbol{u}} \equiv \left(w_x \frac{\partial}{\partial x} + w_y \frac{\partial}{\partial y} + w_z \frac{\partial}{\partial z}\right) (u_x, u_y, u_z) =$$

$$= (\left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right) u_x, \left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right) u_y, \left(\overrightarrow{\boldsymbol{w}} \circ \nabla\right) u_z) =$$

$$= \left(\overrightarrow{\boldsymbol{w}} \circ \operatorname{grad} u_x,\ \overrightarrow{\boldsymbol{w}} \circ \operatorname{grad} u_y,\ \overrightarrow{\boldsymbol{w}} \circ \operatorname{grad} u_z\right) =$$

$$= (w_x\frac{\partial u_x}{\partial x}+w_y\frac{\partial u_x}{\partial y}+w_z\frac{\partial u_x}{\partial z},\ w_x\frac{\partial u_y}{\partial x}+w_y\frac{\partial u_y}{\partial y}+w_z\frac{\partial u_y}{\partial z},$$

$$w_x\frac{\partial u_z}{\partial x}+w_y\frac{\partial u_z}{\partial y}+w_z\frac{\partial u_z}{\partial z}) =$$

$$= w_x\left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial x},\frac{\partial u_z}{\partial x}\right) + w_y\left(\frac{\partial u_x}{\partial y},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}\right) +$$

$$+w_z\left(\frac{\partial u_x}{\partial z},\frac{\partial u_y}{\partial z},\frac{\partial u_z}{\partial z}\right) \equiv$$

$$\equiv w_x\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial x} + w_y\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial y} + w_z\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial z}.$$

gdje je

$$\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial x} \equiv \left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial x},\frac{\partial u_z}{\partial x}\right),$$

$$\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial y} \equiv \left(\frac{\partial u_x}{\partial y},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}\right),$$

$$\frac{\partial \overrightarrow{\boldsymbol{u}}}{\partial z} \equiv \left(\frac{\partial u_x}{\partial z},\frac{\partial u_y}{\partial z},\frac{\partial u_z}{\partial z}\right).$$

Dokaz: Kao i u prethodnom teoremu, sve tvrdnje su izravne posljedice odgovarajućih definicija. Dokažimo npr. tvrdnju 3. Na lijevoj strani je vektorska funkcija

$$\operatorname{rot}\left(\vec{\boldsymbol{w}} \times \vec{\boldsymbol{u}}\right) = \operatorname{rot}(w_y u_z - w_z u_y, w_z u_x - w_x u_z, w_x u_y - w_{y u_x}) =$$

$$\begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ w_y u_z - w_z u_y & w_z u_x - w_x u_z & w_x u_y - w_y u_x \end{vmatrix} =$$

$$\left(\frac{\partial}{\partial y}(w_x u_y - w_y u_x) - \frac{\partial}{\partial z}(w_z u_x - w_x u_z),\right.$$

$$, \frac{\partial}{\partial z}(w_y u_z - w_z u_y) - \frac{\partial}{\partial x}(w_x u_y - w_y u_x),$$

$$\left.\frac{\partial}{\partial x}(w_z u_x - w_x u_z) - \frac{\partial}{\partial y}(w_y u_z - w_z u_y)\right),$$

a desnoj je vektorska funkcija

$$\left(\operatorname{div} \vec{\boldsymbol{u}}\right) \vec{\boldsymbol{w}} - \left(\operatorname{div} \vec{\boldsymbol{w}}\right) \vec{\boldsymbol{u}} + \left(\vec{\boldsymbol{u}} \circ \nabla\right) \vec{\boldsymbol{w}} - \left(\vec{\boldsymbol{w}} \circ \nabla\right) \vec{\boldsymbol{u}} =$$

$$\left(\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)w_x, \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)w_y,\right.$$

$$\left.\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)w_z\right)-$$

$$\left(\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)u_x, \left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)u_y,\right.$$

$$\left.\left(\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}+\frac{\partial w_z}{\partial z}\right)u_z\right)+$$

$$+u_x\left(\frac{\partial w_x}{\partial x},\frac{\partial w_y}{\partial x},\frac{\partial w_z}{\partial x}\right)+u_y\left(\frac{\partial w_x}{\partial y},\frac{\partial w_y}{\partial y},\frac{\partial w_z}{\partial y}\right)$$

$$+u_z\left(\frac{\partial w_x}{\partial z},\frac{\partial w_y}{\partial z},\frac{\partial w_z}{\partial z}\right)-$$

$$-w_x\left(\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial x},\frac{\partial u_z}{\partial x}\right)-w_y\left(\frac{\partial u_x}{\partial y},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}\right)$$

$$-w_z\left(\frac{\partial u_x}{\partial z},\frac{\partial u_y}{\partial z},\frac{\partial u_z}{\partial z}\right).$$

Sređivanjem se pokazuje da se radi o istoj vektorskoj funkciji.

Primijetimo da je tvrdnja 2. isto što i tvrdnja u Teoremu 4.13, 3., te da 5. slijedi iz 4. primjenom Schwarzova teorema.

Napomena Tvrdnja 6. u Teoremu 4.16 smijemo prereći ovako:

$$\operatorname{grad}\operatorname{div}\overrightarrow{\boldsymbol{w}} = \operatorname{rot}\left(\operatorname{rot}\overrightarrow{\boldsymbol{w}}\right) + \Delta\overrightarrow{\boldsymbol{w}},$$

te time dobiti odgovor na pitanje što je gradijent skalarnog polja, koje je divergencija vektorskog polja.

Premda najčešće koristimo pravokutni sustav, u mnogim je slučajevima jednostavnije računati u nekom drugom koordinatnom sustavu.
Zapišimo linearne operatore grad, div, rot i Laplaceov operator Δ u cilindričnom i sfernom sustavu:

U cilindričnomu sustavu $\left(O; \overrightarrow{\boldsymbol{\rho}}_0, \overrightarrow{\boldsymbol{\varphi}}_0, \overrightarrow{\boldsymbol{k}}\right)$:

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\cdot\frac{\partial f}{\partial \varphi}, \frac{\partial f}{\partial z}\right);$$

$$\operatorname{div}\overrightarrow{\boldsymbol{w}} = \frac{1}{\rho}\cdot\frac{\partial(\rho w_\rho)}{\partial \rho} + \frac{1}{\rho}\cdot\frac{\partial w_\varphi}{\partial \varphi} + \frac{\partial w_z}{\partial z};$$

$$\operatorname{rot}\overrightarrow{\boldsymbol{w}}=\left(\frac{1}{\rho}\left(\frac{\partial w_z}{\partial\varphi}-\frac{\partial w_\varphi}{\partial z}\right),\frac{\partial w_\varphi}{\partial z}-\frac{\partial w_z}{\partial\rho},\frac{1}{\rho}\left(\frac{\partial(\rho w_\varphi)}{\partial\rho}-\frac{\partial w_\rho}{\partial\varphi}\right)\right)$$

$$\Delta f=\frac{1}{\rho}\cdot\frac{\partial(\rho\frac{\partial f}{\partial\rho})}{\partial\rho}+\frac{1}{\rho^2}\cdot\frac{\partial^2 f}{\partial\varphi^2}+\frac{\partial^2 f}{\partial z^2}.$$

U sfernomu sustavu $\left(O;\overrightarrow{\boldsymbol{r}}_0,\overrightarrow{\boldsymbol{\theta}}_0,\overrightarrow{\boldsymbol{\varphi}}_0\right)$:

$$\operatorname{grad} f=\left(\frac{\partial f}{\partial r},\frac{1}{r}\cdot\frac{\partial f}{\partial\theta},\frac{1}{r\sin\theta}\cdot\frac{\partial f}{\partial\varphi}\right);$$

$$\operatorname{div}\overrightarrow{\boldsymbol{w}}=\frac{1}{r^2}\cdot\frac{\partial(r^2 w_r)}{\partial r}+\frac{1}{r\sin\theta}\cdot\frac{\partial(w_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\cdot\frac{\partial w_\varphi}{\partial\varphi};$$

$$\operatorname{rot}\overrightarrow{\boldsymbol{w}}=$$

$$\left(\frac{1}{r\sin\theta}\left(\frac{\partial(w_\varphi\sin\theta)}{\partial\theta}-\frac{\partial w_\theta}{\partial\varphi}\right),\frac{1}{r}\left(\frac{1}{\sin\theta}\cdot\frac{\partial w_r}{\partial\varphi}-\frac{\partial(rw_\varphi)}{\partial r}\right)\right.$$

$$\left.,\frac{1}{r}\left(\frac{\partial(rw_\theta)}{\partial r}-\frac{\partial w_r}{\partial\theta}\right)\right)$$

$$\Delta f=\frac{1}{r^2}\cdot\frac{\partial(r^2\frac{\partial f}{\partial r})}{\partial r}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial(\sin\theta\cdot\frac{\partial f}{\partial\theta})}{\partial\theta}+\frac{1}{r^2\sin\theta}\cdot\frac{\partial^2 f}{\partial\varphi^2}.$$

Usmjerena derivacija

Neka je $U : \Omega \to \mathbb{R}$ skalarno polje. Odaberimo bilo koju točku $T_0 \in \Omega$ i bilo koji vektor $\overrightarrow{\boldsymbol{l}} \in \overrightarrow{\mathbb{V}}(T_0)$.

Tada za svaku točku T na zraci l određenoj s $\overrightarrow{\boldsymbol{l}}$ vrijedi

$$\overrightarrow{TT_0} = t\,\overrightarrow{\boldsymbol{l}}_0,$$

za neki $t \in [0, \infty\rangle$, pri čemu je $\overrightarrow{\boldsymbol{l}}_0 = \frac{\overrightarrow{\boldsymbol{l}}}{\|\overrightarrow{\boldsymbol{l}}\|}$ pripadni jedinični vektor. Broj t je, zapravo, udaljenost $d(T_0, T)$ od T_0 do T.

Prosječnom promjenom skalarnoga polja U **od točke** T_0 **do točke** T nazivamo količnik

$$\frac{U(T) - U(T_0)}{d(T_0, T)} = \frac{U(T) - U(T_0)}{t} \in \mathbb{R}, \quad \overrightarrow{TT_0} = t\,\overrightarrow{\boldsymbol{l}}_0.$$

Pokazuje se ovdje važnim istražiti granični slučaj $T \to T_0$ po l, tj. $t \to 0$. Da bi navedeni količnik (u graničnom slučaju) imao smisla za svaki izbor jediničnog vektora $\overrightarrow{\boldsymbol{l}}_0$ nužno je da područje Ω bude ”lokalno konveksno”. Dakako, dovoljno je da $\Omega \equiv D \subseteq \mathbb{R}^3$ bude otvoren skup.

Definicija 4.17 Neka je $U : \Omega \to \mathbb{R}$ skalarno polje. Graničnu vrijednost (ako postoji)

$$\lim_{t \to 0} \frac{U(T) - U(T_0)}{t} \in \mathbb{R}, \quad \overrightarrow{TT_0} = t \overrightarrow{\boldsymbol{l}}_0,$$

nazivamo **derivacijom skalarnoga polja U u točki T_0 u smjeru $\overrightarrow{l}$** (ili, kraće, **usmjerenom** (**skalarnom**) **derivacijom**) i označujemo s

$$\frac{\partial U(T_0)}{\partial \overrightarrow{\boldsymbol{l}}}.$$

U praktičnom računu je puno jednostavnije baratati predstavnikom $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$, skalarnoga polja U u koordinatnomu sustavu $\left(O; \overrightarrow{\boldsymbol{i}}, \overrightarrow{\boldsymbol{j}}, \overrightarrow{\boldsymbol{k}}\right)$.

Prvo, točkama $T_0 = (x_0, y_0, z_0)$ i $T = (x, y, z)$ pripadaju radijus-vektori $\overrightarrow{OT_0} = x_0 \overrightarrow{\boldsymbol{i}} + y_0 \overrightarrow{\boldsymbol{j}} + z_0 \overrightarrow{\boldsymbol{k}}$ i $\overrightarrow{OT} = x \overrightarrow{\boldsymbol{i}} + y \overrightarrow{\boldsymbol{j}} + z \overrightarrow{\boldsymbol{k}}$. Nadalje, $\overrightarrow{OT} = \overrightarrow{OT_0} + \overrightarrow{T_0T} = \overrightarrow{OT_0} + t \overrightarrow{\boldsymbol{l}}_0$, dakle,

$$x \overrightarrow{\boldsymbol{i}} + y \overrightarrow{\boldsymbol{j}} + z \overrightarrow{\boldsymbol{k}} = \left(x_0 + t \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{i}}\right)\right) \overrightarrow{\boldsymbol{i}} +$$

$$\left(y_0 + t \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{j}}\right)\right) \overrightarrow{\boldsymbol{j}} + \left(z_0 + t \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{k}}\right)\right) \overrightarrow{\boldsymbol{k}}.$$

Slijedi da se koordinate svake točke T na zraci što ju određuju T_0 i $\overrightarrow{\boldsymbol{l}}_0$ opisuju linearnim jednadžbama

$$x = x_0 + \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{i}}\right) t, \quad y = y_0 + \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{j}}\right) t,$$

$$z = z_0 + \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{k}}\right) t, \quad t \in [0, \infty\rangle .$$

<u>Uočimo:</u> Ovdje je

$$\overrightarrow{\boldsymbol{l}}_0 = \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{i}}\right) \overrightarrow{\boldsymbol{i}} + \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{j}}\right) \overrightarrow{\boldsymbol{j}} + \left(\overrightarrow{\boldsymbol{l}}_0 \circ \overrightarrow{\boldsymbol{k}}\right) \overrightarrow{\boldsymbol{k}} =$$

$$= \cos\alpha \, \overrightarrow{\boldsymbol{i}} + \cos\beta \, \overrightarrow{\boldsymbol{j}} + \cos\gamma \, \overrightarrow{\boldsymbol{k}},$$

gdje je $\alpha = \measuredangle\left(\overrightarrow{\boldsymbol{l}}_0, \overrightarrow{\boldsymbol{i}}\right)$, $\beta = \measuredangle\left(\overrightarrow{\boldsymbol{l}}_0, \overrightarrow{\boldsymbol{j}}\right)$, $\gamma = \measuredangle\left(\overrightarrow{\boldsymbol{l}}_0, \overrightarrow{\boldsymbol{k}}\right)$.

Proizlazi da je prije definirana usmjerena derivacija, zapravo, derivacija funkcijske kompozicije

$$[0,\infty\rangle \overset{\vec{\boldsymbol{w}}=(w_x,w_y,w_z)}{\longrightarrow} D \xrightarrow{f} \mathbb{R}, \quad \text{tj.}$$

$$t \mapsto \vec{\boldsymbol{w}}(t) = (x_0 + \left(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{i}}\right) t,$$

$$y_0 + \left(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{j}}\right) t, z_0 + \left(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{k}}\right) t) \mapsto f(\vec{\boldsymbol{w}}(t)),$$

u točki $t = 0$. Prema tomu,

$$\frac{\partial U(T_0)}{\partial \vec{\boldsymbol{l}}} \equiv \frac{\partial f(x_0,y_0,z_0)}{\partial \vec{\boldsymbol{l}}} = \left(f \circ \vec{\boldsymbol{w}}\right)'(0) =$$

$$\frac{\partial f(\vec{\boldsymbol{w}}(0))}{\partial x} \cdot w_x'(0) + \frac{\partial f(\vec{\boldsymbol{w}}(0))}{\partial y} \cdot w_y'(0) + \frac{\partial f(\vec{\boldsymbol{w}}(0))}{\partial z} \cdot w_z'(0) =$$

$$\frac{\partial f(x_0,y_0,z_0)}{\partial x}(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{i}}) + \frac{\partial f(x_0,y_0,z_0)}{\partial y}(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{j}}) +$$

$$+\frac{\partial f(x_0,y_0,z_0)}{\partial z})(\vec{\boldsymbol{l}}_0 \circ \vec{\boldsymbol{k}}) = (\operatorname{grad} f(x_0,y_0,z_0)) \circ \vec{\boldsymbol{l}}_0.$$

Time smo dokazali ovaj teorem:

Teorem 4.18 Derivacija skalarnog polja U, zadanoga funkcijom $f : D \to \mathbb{R}$, u točki $T_0 = (x_0, y_0, z_0)$ u smjeru $\overrightarrow{\boldsymbol{l}}$ *($\neq \overrightarrow{\mathbf{0}}$)* jednaka je skalarnom umnošku pripadnoga gradijenta s jediničnim vektorom $\overrightarrow{\boldsymbol{l}}_0$, tj. projekciji vektora $\operatorname{grad} f(x_0, y_0, z_0)$ na vektor $\overrightarrow{\boldsymbol{l}}$:

$$\frac{\partial f(x_0, y_0, z_0)}{\partial \overrightarrow{\boldsymbol{l}}} = (\operatorname{grad} f(x_0, y_0, z_0)) \circ \overrightarrow{\boldsymbol{l}}_0.$$

Primjer 13 Izračunajmo derivaciju skalarnoga polja

$$(x, y, z) \mapsto f(x, y, z) = x^2 + 3yz + 5$$

u točki $T = (3, 2, -1)$ u smjeru $\overrightarrow{\boldsymbol{l}} = \overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + \overrightarrow{\boldsymbol{k}}$.

Po Teoremu 4.18, iz

$$\operatorname{grad} f(3, 2, -1) = (2x, 3z, 3y)|_{(3,2,-1)} = (6, -3, 6), \quad \text{i}$$

$$\overrightarrow{\boldsymbol{l}}_0 = \frac{\overrightarrow{\boldsymbol{l}}}{||\overrightarrow{\boldsymbol{l}}||} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

dobivamo da je tražena usmjerena derivacija $\frac{\partial f(3,2,-1)}{\partial \overrightarrow{\boldsymbol{l}}}$ jednaka

$$\frac{\partial f(3,2,-1)}{\partial \overrightarrow{\boldsymbol{l}}} = (\operatorname{grad} f(3,2,-1)) \circ \overrightarrow{\boldsymbol{l}}_0 =$$

$$= (6,-3,6) \circ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 3\sqrt{3}.$$

Iz Teorema 4.18 slijedi da skalarno polje f najbrže raste u smjeru što ga određuje $\operatorname{grad} f$, odnosno, da najbrže pada u smjeru što ga određuje $-\operatorname{grad} f$. Drugim riječima, $\operatorname{grad} f$ **pokazuje smjer najbrže promjene** skalarnoga polja f. Štoviše, gradijent se time može i okarakterizirati.

Teorem 4.19 Neka funkcija $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$, opisuje diferencijabilno skalarno polje U. Tada gradijent toga polja u svakoj točki $T_0 = (x_0, y_0, z_0)$ ima za pravac okomicu na razinsku plohu u T_0, a iznos mu je jednak apsolutnoj vrijednosti pripadne usmjerene derivacije. Sažeto,

$$\operatorname{grad} f(x_0, y_0, z_0) = \frac{\partial f(x_0, y_0, z_0)}{\partial \overrightarrow{\boldsymbol{n}}} \overrightarrow{\boldsymbol{n}}_0,$$

pri čemu je $\overrightarrow{\boldsymbol{n}}$ normala na razinsku plohu $f(x, y, z) = f(x_0, y_0, z_0)$.

Dokaz: Odaberimo bilo koju točku $T_0 = (x_0, y_0, z_0) \in D$. Jednadžba pripadne razinske plohe $U(T) = c \equiv U(T_0)$ prelazi u $f(x, y, z) = f(x_0, y_0, z_0)$, što povlači

$$0 = df(x, y, z) =$$

$$\frac{\partial f(x_0, y_0, z_0)}{\partial x} dx + \frac{\partial f(x_0, y_0, z_0)}{\partial y} dy + \frac{\partial f(x_0, y_0, z_0)}{\partial z} dz =$$

$$= (\operatorname{grad} f(x_0, y_0, z_0)) \circ d\overrightarrow{\boldsymbol{r}}),$$

pri čemu $d\overrightarrow{\boldsymbol{r}}$ označuje vektor $dx\overrightarrow{\boldsymbol{i}} + dy\overrightarrow{\boldsymbol{j}} + dz\overrightarrow{\boldsymbol{k}} \equiv (dx, dy, dz)$.

Budući da pratimo zbivanje na razinskoj plohi u točki T_0, to $d\overrightarrow{\boldsymbol{r}}$ smijemo tumačiti kao infinitezimalni pomak u njezinoj tangencijalnoj ravnini točkom T_0. Prema tomu,

$$(\operatorname{grad} f(x_0, y_0, z_0)) \circ d\overrightarrow{\boldsymbol{r}} = 0$$

znači da je $\operatorname{grad} f(x_0, y_0, z_0)$ okomit na tu tagencijalnu ravninu, tj. na promatranu razinsku plohu u točki T_0, dakle, usporedan s pripadnim normalnim vektorom $\overrightarrow{\boldsymbol{n}}$. Slijedi $\operatorname{grad} f(x_0, y_0, z_0) = \lambda \overrightarrow{\boldsymbol{n}}_0$ pa je

$$\lambda = (\operatorname{grad} f(x_0, y_0, z_0)) \circ \overrightarrow{\boldsymbol{n}}_0.$$

Po Teoremu 4.18 je

$$(\operatorname{grad} f(x_0, y_0, z_0)) \circ \overrightarrow{\boldsymbol{n}}_0 = \frac{\partial f(x_0, y_0, z_0)}{\partial \overrightarrow{\boldsymbol{n}}},$$

što onda daje

$$\operatorname{grad} f(x_0, y_0, z_0) = \frac{\partial f(x_0, y_0, z_0)}{\partial \overrightarrow{\boldsymbol{n}}} \overrightarrow{\boldsymbol{n}}_0.$$

Napomena Pomoću Teorema 4.18 je lako izračunati jedinični normalni vektor na razinsku plohu skalarnog polja. Naime, za svaku točku $T_0 = (x_0, y_0, z_0)$, vrijedi

$$\vec{\boldsymbol{n}}_0 = \frac{\operatorname{grad} f(x_0, y_0, z_0)}{\|\operatorname{grad} f(x_0, y_0, z_0)\|}.$$

Primjer 14 Izračunajmo jedinični normalni vektor na plohu $z = xy$ u točki $T_0 = (2, 3, 6)$.

Smijemo pretpostaviti da je promatrana ploha $z = xy$ neka razinska ploha $f(x, y, z) = c$ nekog skalarnog polja $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$. Tako dobivamo

$$f(x, y, z) = xy - z + c,$$

pa je

$$\operatorname{grad} f(2, 3, 6) = (y, x, -1)|_{(2,3,6)} = (3, 2, -1)$$

i, napokon,

$$\vec{\boldsymbol{n}}_0(T_0) = \frac{\operatorname{grad} f(2, 3, 6)}{\|\operatorname{grad} f(2, 3, 6)\|} = \frac{1}{\sqrt{14}} \left(3\,\vec{\boldsymbol{i}} + 2\,\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}} \right).$$

Razmotrimo sada kako bi se usmjerena derivacija mogla osmisliti u vektorskom polju. Neka funkcija $\overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, predstavlja vektorsko polje $\overrightarrow{\boldsymbol{V}}$. Promatrajmo opet čvrstu točku $T_0 = (x_0, y_0, z_0)$, varijabilnu točku $T = (x, y, z)$ i količnik

$$\frac{\overrightarrow{\boldsymbol{w}}(x, y, z) - \overrightarrow{\boldsymbol{w}}(x_0, y_0, z_0)}{t} \in \mathbb{R}^3,$$

koji opisuje **prosječnu promjenu vektorskoga polja** $\overrightarrow{\boldsymbol{V}} \equiv \overrightarrow{\boldsymbol{w}}$ (pri prijelazu iz T_0 u T) u smjeru $\overrightarrow{\boldsymbol{l}} \equiv \overrightarrow{T_0T} = t\,\overrightarrow{\boldsymbol{l}}_0$, pri čemu je dužina $\overline{T_0T} \subseteq D$.

Definicija 4.20 Neka funkcija $\overrightarrow{\boldsymbol{w}} : D \to \mathbb{R}^3$, $D \subseteq \mathbb{R}^3$, predstavlja vektorsko polje $\overrightarrow{\boldsymbol{V}}$, te neka su $T_0 = (x_0, y_0, z_0), T = (x, y, z) \in D$, $\overline{T_0T} \subseteq D$ i $\overrightarrow{T_0T} = t\,\overrightarrow{\boldsymbol{l}}_0$. Graničnu vrijednost (ako postoji)

$$\lim_{t \to 0} \frac{\overrightarrow{\boldsymbol{w}}(x, y, z) - \overrightarrow{\boldsymbol{w}}(x_0, y_0, z_0)}{t} \in \mathbb{R}^3,$$

nazivamo **derivacijom vektorskoga polja $\overrightarrow{\boldsymbol{V}}$ *(*$\overrightarrow{\boldsymbol{w}}$*)* u točki T_0 u smjeru $\overrightarrow{\boldsymbol{l}}$** *(*ili, kraće, **usmjerenom (vektorskom) derivacijom**) i označujemo s

$$\frac{\partial \overrightarrow{\mathbf{V}}(T_0)}{\partial \overrightarrow{\boldsymbol{l}}} \equiv \frac{\partial \overrightarrow{\boldsymbol{w}}(x_0, y_0.z_0)}{\partial \overrightarrow{\boldsymbol{l}}}.$$

Teorem 4.21 Derivacija vektorskog polja $\overrightarrow{\boldsymbol{w}}$ u točki $T_0 = (x_0, y_0, z_0)$ u smjeru $\overrightarrow{\boldsymbol{l}}$ ($\neq \mathbf{0}$), ako postoji, jednaka je vrijednosti operatora $\left(\overrightarrow{\boldsymbol{l}}_0 \circ \nabla\right)$ primijenjenog na $\overrightarrow{\boldsymbol{w}}$ u (x_0, y_0, z_0), tj.

$$\frac{\partial \overrightarrow{\boldsymbol{w}}(x_0, y_0.z_0)}{\partial \overrightarrow{\boldsymbol{l}}} = \left(\left(\overrightarrow{\boldsymbol{l}}_0 \circ \nabla\right) \overrightarrow{\boldsymbol{w}}\right)(x_0, y_0, z_0) =$$

$$= \left(\overrightarrow{\boldsymbol{l}}_0 \circ \operatorname{grad} w_x,\ \overrightarrow{\boldsymbol{l}}_0 \circ \operatorname{grad} w_y,\ \overrightarrow{\boldsymbol{l}}_0 \circ \operatorname{grad} w_z\right)_{(x_0,y_0,z_0)} =$$

$$= \left(\cos\alpha \frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial x} + \cos\beta \frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial y} + \cos\gamma \frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial z}\right)_{(x_0,y_0,z_0)}$$

gdje su $\cos\alpha$, $\cos\beta$ i $\cos\gamma$ smjerovni kosinusi od $\overrightarrow{\boldsymbol{l}}$, tj. komponenete jediničnoga vektora $\overrightarrow{\boldsymbol{l}}_0$.

(Operator

$$u \circ \nabla = u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}$$

smo uveli u Teoremu 4.16)

Primjer 15 Izračunajmo derivaciju vektorskoga polja

$$(x,y,z) \mapsto \overrightarrow{\boldsymbol{w}}(x,y,z) = (yz, zx, xy)$$

u točki $T_0 = (1, -\frac{1}{2}, 2)$ u smjeru $\overrightarrow{\boldsymbol{l}} = 2\,\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} - 2\,\overrightarrow{\boldsymbol{k}}$.

Budući da je $\overrightarrow{\boldsymbol{l}}_0 = \frac{1}{3}(2\,\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} - 2\,\overrightarrow{\boldsymbol{k}})$, po Teoremu 4.21 dobivamo

$$\frac{\partial \overrightarrow{\boldsymbol{w}}(1, -\frac{1}{2}, 2)}{\partial \overrightarrow{\boldsymbol{l}}} = \left(\left(\overrightarrow{\boldsymbol{l}}_0 \circ \nabla\right) \overrightarrow{\boldsymbol{w}}\right)(1, -\tfrac{1}{2}, 2) =$$

$$\left(\frac{2}{3}\frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial x} + \frac{1}{3}\frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial y} - \frac{2}{3}\frac{\partial \overrightarrow{\boldsymbol{w}}}{\partial z}\right)_{(1,-\frac{1}{2},2)} =$$

$$\left[\frac{2}{3}(0,z,y) + \frac{1}{3}(z,0,x) - \frac{2}{3}(y,x,0)\right]_{(1,-\frac{1}{2},2)} =$$

$$\frac{1}{3}\,(z-2y, 2z-2x, 2y+x)_{(1,-\frac{1}{2},2)} = \overrightarrow{\boldsymbol{i}} + \frac{2}{3}\overrightarrow{\boldsymbol{j}}\,.$$